Problems in the Numerical Simulation of Models with Heterogeneous Agents and Economic Distortions*

Adrian Peralta-Alva
Research Division
Federal Reserve Bank of Saint Louis

Manuel S. Santos
Department of Economics
University of Miami

Abstract

Our work has been concerned with the numerical simulation of dynamic economies with heterogeneous agents and economic distortions. Recent research has drawn attention to inherent difficulties in the computation of competitive equilibria for these economies: A continuous Markovian solution may fail to exist, and some commonly used numerical algorithms may not deliver accurate approximations. We consider a reliable algorithm set forth in Feng et al. (2009), and discuss problems related to the existence and computation of Markovian equilibria, as well as convergence and accuracy properties. We offer new insights into numerical simulation.

JEL classification numbers: C6, D5, E2.

Keywords: Heterogeneous agents, taxes, externalities, financial frictions, competitive equilibrium, computation, simulation.

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*We are grateful to Felix Kubler and Karl Schmedders for helpful remarks, and to Chanont Banternghansa for his excellent research assistance. The views expressed are those of the individual authors and do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors. This paper was written while Manuel Santos was visiting Universidad Carlos III de Madrid, and was partially funded under projects SEJ2005-05831 and ECO2008-04073 of the Spanish Ministry of Education, and Catedra de Excelencia of Banco Santander.

1 Introduction

In this paper we review some fundamental issues in the computation and numerical simulation of dynamic economic models with heterogeneous agents and market distortions (e.g., externalities, distortionary taxation and financial frictions). These models are now an integral part of modern macroeconomics and finance, and play a central role in the analysis of fiscal and monetary policies, social security systems and savings over the life cycle, and the determinants of asset price volatility and interest rates.

A main problem with these models is numerical tractability. Economists use numerical simulations to get quantitative predictions. But for models with economic distortions or with an infinite number of overlapping generations, their equilibrium solutions cannot be computed by associated global optimization problems as the welfare theorems do not hold. In the presence of multiple competitive equilibria, there is a problem of coordinating expectations that may result in the inability to invoke Bellman's optimality principle.[1] We then have that a Markovian representation of equilibria may only be possible when conditioning over an expanded set of state variables; moreover, this generalized Markovian law of motion may not be a continuous mapping. Therefore, numerical dynamic programming algorithms and other well known numerical procedures – which search for a continuous policy function – have a limited application for the computation of competitive equilibria of these economies.

These technical issues are mostly ignored in the applied literature, but are crucial to guarantee that a numerical algorithm provides a sufficiently good approximation of the postulated economic model so that we can make reasonable inferences form the output of our computations. Of course, reliable techniques are computationally costly and may not always be feasible, but they should be the starting point for the construction of faster methods with good convergence and accuracy properties.

Examples of non-existence of Markov equilibria can be found in Kubler and Schmedders (2002), Kubler and Polemarchakis (2004) and Santos (2002). Following Duffie et al. (1994), the existence of a Markov equilibrium in a generalized space of variables is proved in Kubler and Schmedders (2003) for an asset pricing model with collateral constraints. Feng et al. (2009) extend these existence results to other economies, and define a Markov equilibrium as a solution over an expanded state of variables that includes the shadow values of investment. The addition of the shadow values of investment as state variables was originally proposed by Kydland and

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[1] For a good discussion of these issues see the early seminal papers of Hellwig (1983) and Kydland and Prescott (1980).

Prescott (1980), and later used in Marcet and Marimon (1998) for recursive contracts, and in Phelan and Stacchetti (2001) for a competitive economy with a representative agent. The main insight of Feng et al. (2009) is to apply this extension of the state space to a reliable algorithm along the lines of Kubler and Schmedders (2003) for the computation of competitive equilibrium models with heterogeneous agents. This redefinition of the state space may be quite effective in the computation of these models. To guarantee convergence to a fixed-point solution the equilibrium operator iterates over a decreasing sequence of candidate compact equilibrium correspondences rather then over functions since no equilibrium function may be continuous. Feng et al. (2009) propose a discretized numerical algorithm that is shown to have good convergence and accuracy properties. This constitutes the first attempt to build numerical foundations for this type of algorithm in a context in which no equilibrium selection may be continuous.

We would like to remark that our algorithm is also in the spirit of Abreu, Pearce, and Stacchetti (1990) but there are two major differences. First, we will be concerned with developing approximation properties of a discretized algorithm for the simulation of competitive models with heterogeneous agents and market distortions rather than solving for equilibria of super games. Second, their state space includes a vector of continuation utilities, which is not a useful state variable for our purposes.

To illustrate some of the pitfalls of using non-reliable algorithms, we also provide an example in which a regular computational method may actually converge to a wrong fixed-point solution. In Feng et al. (2009) we consider another example of an overlapping generations economy with chaotic dynamics where our algorithm would work well, but more common algorithms may deliver misleading results. For more complex models with a large number of state variables [e.g., Krusell and Smith (1998)] our algorithm may not be computationally feasible, but it is our experience that some other heuristic extensions of standard algorithms may display sizable computational errors.

Finally, we should emphasize that our objective is to compute the set of sequential competitive equilibria, and this set need not be convex. Arbitrary randomizations over the equilibrium correspondence may lead to a drastic expansion of the fixed-point solution. Hence, we compute equilibria directly rather than convex hulls of candidate equilibrium correspondences. Extending the techniques on convex correspondences of Judd, Yeltekin and Conklin (2003) seems to be a challenging task of rather limited applicability. Our methods could also be applied to the computation of sunspot equilbria. Our operator obtains the set of all equilibria, and therefore it seems possible to compute the set of all sunspot equilibria. But our discussion here will be limited to the computation of standard sequential competitive equilibria.

2 The Analytical Framework

Time is discrete, $t = 0, 1, 2, \cdots$. The state of the economy includes a vector of endogenous variables x and vector of exogenous shocks z. Vector x belongs to a compact domain X and contains all predetermined variables, such as agents' holdings of physical capital, human capital, and financial assets. The exogenous state vector follows a Markov chain $(z_t)_{t\geq 0}$ over a finite set Z. This Markovian process is described by positive transition probabilities $\pi(z'|z)$ for all $z, z' \in Z$. The initial state, $z_0 \in Z$, is known to all agents in the economy. Then $z^t = (z_1, z_2, ..., z_t) \in Z^t$ is a history of shocks, often called a date-event or node. Let y denote the vector of all other endogenous variables. These variables could be equilibrium prices or choice variables such as consumption and investment.

In various economic models the dynamics of the state vector x is governed by a system of non-linear equations:

$$\varphi(x_{t+1}, x_t, y_t, z_t) = 0. \tag{1}$$

Function φ may incorporate technological constraints and individual budget constraints. Let m denote a vector of shadow values of the marginal return to investment for all assets and all agents. This vector is supposed to lie in a compact space M, and it will be a function of existing variables such as prices, rates of interest, and marginal utilities and productivities:

$$m_t = h(x_t, y_t, z_t). \tag{2}$$

Let us assume that a sequential competitive equilibrium exists and can be represented by a sequence $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ satisfying (1)-(2), and the additional system of equations

$$\Phi(x_t, y_t, z_t, E_t[m_{t+1}]) = 0, \tag{3}$$

where $E[m]$ is an expectations operator that conditions on state z_t. Function Φ may describe individual optimality conditions (such as Euler equations), market-clearing conditions, and various types of restrictions such as short-sales and liquidity requirements.[2] We consider that equations (1)-(3)

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[2] Note that inequalities can always be transformed into equalities by a judicious choice

fully characterize a sequential competitive equilibrium, and that φ, h, and Φ are continuous functions.

To compute the set of sequential competitive equilibria we define the Markovian equilibrium correspondence $V^*(x,z)$ containing all the equilibrium vectors m for any given state (x,z). From this correspondence V^*, we can generate recursively the set of all equilibria as V^* is the fixed point of an operator $B : V \longmapsto B(V)$ that links state variables to future equilibrium states. Operator B embodies all equilibrium conditions such as agents' optimization and market-clearing conditions from any initial node z to all immediate successor states z_+. More precisely, let $B(V)(x,z)$ be the set of all values $m = h(x,y,z)$ with the property: For given x, z there exist y and $m_+(z_+) \in V(x_+, z_+)$ with $z_+ \in Z$ such that

$$\Phi(x,y,z,E[m_+(z_+)]) = 0,$$

and

$$\varphi(x_+, x, y, z) = 0.$$

Therefore, we are certain that for each $m \in B(V)(x,z)$ there are continuation values that satisfy the temporary equilibrium conditions. The following result is proved in Feng et al. (2009):

Theorem 2.1 *(convergence) Let V_0 be a compact-valued correspondence such that $V_0 \supset V^*$. Let $V_n = B(V_{n-1}), n \geq 1$. Then, $V_n \to V^*$ as $n \to \infty$. Moreover, V^* is the largest fixed point of the operator B, that is, if $V = B(V)$, then $V \subset V^*$.*

Theorem 2.1 provides the theoretical foundations of our algorithm since we can apply operator B to any large compact set $V_0(x,z) \supset V^*(x,z)$ and then iterate until a desirable level of convergence is attained. An important advantage of our approach is that if multiple equilibria exist, we can find all of them. Now, from operator $B : graph(V^*) \to graph(V^*)$ we can select a measurable policy function $y = g^y(x,z,m)$, and a transition function $m_+(z_+) = g^m(x,z,m;z_+)$, for all $z_+ \in Z$. These functions give a Markovian characterization of a dynamic equilibrium in the enlarged state space.

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of additional variables.

3 A Growth Model with Taxes

The economy is made up of a representative household and a single firm. For a given sequence of interest rates $\{r_t\}$ and taxes $\{\tau_t\}$ the representative household solves the following optimization problem

$$\max \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

$$\begin{aligned} & s.t. \\ & c_t + k_{t+1} \leq \pi_t + (1-\tau_t)\, r_t k_t + T_t \\ & k_0 \text{ given, } 0 < \beta < 1, \\ & c_t \geq 0,\ k_{t+1} \geq 0 \text{ for all } t \geq 0. \end{aligned}$$

Here, c_t denotes consumption, k_t is the individual capital holdings. Taxes on capital income $\{\tau_t\}$ are functions of the aggregate capital stock K_t. All tax revenues are rebated back to the consumer as lump-sum transfers T_t.

The representative firm seeks to maximize one-period profits by employing the optimal amount of capital

$$\pi_t = \max_{K_t}\ f(K_t) - r_t K_t.$$

Let

$$f(K) = K^{1/3},\ \beta = 0.95.$$

We then consider the following piecewise linear tax schedule on capital rents:

$$\tau(K) = \begin{cases} 0.10 & \text{if } K \leq 0.160002 \\ 0.05 - 10(K - 0.165002) & \text{if } 0.160002 \leq K \leq 0.170002 \\ 0 & \text{if } K \geq 0.170002. \end{cases}$$

This example is particularly attractive since it can be easily computed by our algorithm. Santos (2002, Prop. 3.4) shows that a continuous Markov equilibrium fails to exist for this specification of the model. There are three steady states, the middle one is unstable and has two complex eigenvalues, while the other two steady states are saddle-path stable.

Standard algorithms for optimization problems approximating the Euler equation would solve for a continuous policy function of the form

$$k_{t+1} = g(k_t, \xi),$$

where g belongs to a finite dimensional space of continuous functions as defined by a vector of parameters ξ. We obtain an estimate for ξ by forming a discrete system of Euler equations over as many grid points k^i as the dimensionality of the parameter space:

$$u'(k^i, g(k^i, \xi)) = \beta u'(g(k^i, \xi), g(g(k^i, \xi), \xi)) \cdot \left[f'(g(k^i, \xi))(1 - \tau(g(k^i, \xi))) \right].$$

We assume that $g(k^i, \xi)$ belongs to the class of piecewise linear functions, and employ a uniform grid of 5000 points over the domain $k \in [0.14..0.19]$. The resulting approximation, together with the highly accurate solution using our algorithm are illustrated in Figure 1.

This piecewise linear approximation of the Euler equation over piecewise continuous functions converged up to computer precision in only 3 iterations. This fast convergence of the numerical method is actually deceptive because as pointed out above no continuous policy function does exist. A further test of the fixed-point solution of this algorithm based on the Euler equation residuals produced mixed results. First, the average Euler equation residual (a standard accuracy measure) over the domain of feasible capitals is fairly small, i.e. it is equal to 0.0073 as illustrated in Figure 2 below. Second, the maximum Euler equation residual is slightly more pronounced in a small area near the unstable steady state. But even in that area, the magnitude of the error is not extremely large: In three tiny intervals the Euler equation residuals are just around 0.06.

Of course, the dynamic behavior implied by the continuous function approximation is quite different from the true one as it displays four more steady states, and changes substantially the basins of attraction of the original steady states (see Figure 1). Therefore, from these computational tests a researcher may be led to conclude that the purported continuous policy function should mimic well the true equilibrium dynamics.

4 Numerical Implementation

We first partition the state space into a finite set of simplices $\left\{X^j\right\}$ with non-empty interior and maximum diameter h. Over this partition we define a family of step correspondences that take constant values over each X^j. To obtain a computer representation of a step correspondence we resort to an outer approximation in which each set-value is defined by N elements. Using these two discretizations we obtain a computable approximation of operator B, which we denote by $B^{h,N}$. By a suitable selection of an initial condition V_0 and of these outer approximations, the sequence $\{V_{n+1}^{h,N}\}$ defined recursively as $V_{n+1}^{h,N} = B^{h,N} V_n^{h,N}$ converges to a limit point $V^{*,h,N}$, which must contain the equilibrium correspondence V^*. The following result is proved in Feng et al. (2009):

Theorem 4.1 *For given h, N, and initial condition $V_0 \supseteq V^*$, consider the recursive sequence $\{V_{n+1}^{h,N}\}$ defined as $V_{n+1}^{h,N} = B^{h,N}V_n^{h,N}$. Then, (i) $V_n^{h,N} \supseteq V^*$ for all n; (ii) $V_n^{h,N} \to V^{*,h,N}$ uniformly as $n \to \infty$; and (iii) $V^{*,h,N} \to V^*$ as $h \to 0$ and $N \to \infty$.*

To assess model predictions, analysts usually calculate moments of the simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ from a numerical approximation. The idea is that the simulated moments should approach those obtained from the original model. Assuming that the optimal policy is a continuous function, Santos and Peralta-Alva (2005) establish various convergence properties of the simulated moments. They also provide examples of non-existence of stochastic steady-state solutions for non-continuous functions, and lack of convergence of empirical distributions to some invariant distribution of the model. We now briefly outline a reliable simulation procedure that circumvents the lack of continuity of the equilibrium law of motion. We are just advancing arguments from our new paper [Santos and Peralta-Alva (2009)]. We summarize these arguments as follows:

(i) Simulation of the computed equilibrium laws of motion $y = g_n^{y,h,N}(x,z,m)$, *and* $m_+(z_+) = g_n^{m,h,N}(x,z,m;z_+)$. From these approximate equilibrium functions we can generate simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$. We prove that there are tight upper USM and lower LSM bounds such that with probability one the corresponding moments from simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ stay within the prescribed bounds. More precisely, let $s = (x,y,m)$ and $f : S \times Z \to R_+$ be a function of interest. Let $\left(\sum_{t=0}^{T} f(s_t,z_t)\right)/T$ represents a simulated moment or some other statistic. Then, with probability one, every limit point of $\left(\sum_{t=0}^{T} f(s_t,z_t)\right)/T$ must be within the corresponding bounds LSM and USM.

(ii) Existence of an invariant distribution for the original model: An invariant distribution may actually not exist. One way to guarantee existence is to convexify the equilibrium correspondence. Thus, following Blume (1982) and Duffie et al. (1994) we randomize over continuation values of operator B. Following these papers, we construct a new operator B^{cv} that is a convex-valued correspondence in the space of probability measures. This correspondence has an invariant distribution $\mu^* \in B^{cv}(\mu^*)$.

(iii) Accuracy of the simulated moments: For every $\epsilon > 0$ we can consider a sufficiently good discretized operator $B^{h,N}$ and equilibrium correspondence $V_n^{h,N}$ such that for every simulated path $(s_t,z_t)_{t=0}^{\infty}$ there are in-

variant distributions μ^*, μ'^* of B^{cv} satisfying $\int f(s,z)d\mu^* - \epsilon \leq \left(\sum_{t=0}^{T} f(s_t, z_t)\right)/T \leq \int f(s,z)d\mu'^* + \epsilon$ almost surely. Therefore, for a sufficiently fine approximation the moments from simulated paths are close to the set of moments of the invariant distributions of the model. Of course, if B^{cv} has a unique invariant distribution μ^* then $\mu'^* = \mu^*$ and the above expression reads as $\int f(s,z)d\mu^* - \epsilon \leq \left(\sum_{t=0}^{T} f(s_t, z_t)\right)/T \leq \int f(s,z)d\mu^* + \epsilon$.

Duffie et *al*. (1994) argue that operator B^{cv} allows for some form of sunspot equilibria since the randomization proceeds over equilibrium distributions rather than over an extraneous sunspot variable. This is not, however, the interpretation that we want to give to our simulations. For us, the primitive elements in our analysis are the Markovian equilibrium functions which are obtained from the original equilibrium correspondences without performing arbitrary randomizations. We are therefore simulating the true model. At a later stage, to bound the range of variation of the simulated moments for both the approximate and true solutions we consider the invariant distributions of the convex-valued operator B^{cv}. These bounds may not be tight when the regularized operator B^{cv} expands substantially the stochastic dynamics.

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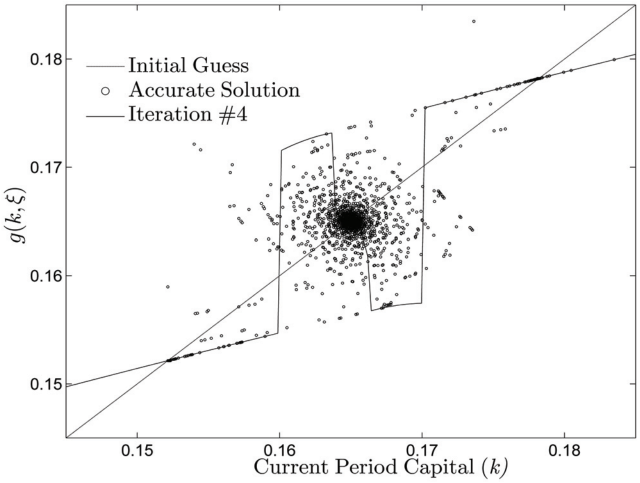

Figure 1: Accurate Solution vs. Continuous Policy Approximation.

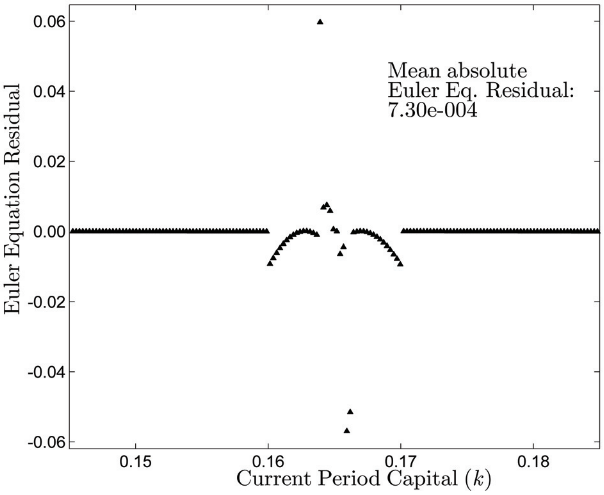

Figure 2: Euler Equation Residual of Continuous Approximation.